New exact solutions for power–law inflation Friedmann models

Hans-Jürgen Schmidt

Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10
D-14469 Potsdam, Germany, E-mail: hjschmi@rz.uni-potsdam.de

Abstract

We consider the spatially flat Friedmann model

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,.$$

For $a \approx t^p$, especially, if $p \geq 1$, this is called power-law inflation. For the Lagrangian $L = R^m$ with $p = -(m-1)(2m-1)/(m-2)$ power-law inflation is an exact solution, as it is for Einstein gravity with a minimally coupled scalar field Φ in an exponential potential $V(\Phi) = \exp(\mu\Phi)$ and also for the higher-dimensional Einstein equation with a special Kaluza-Klein ansatz. The synchronized coordinates are not adapted to allow a closed-form solution, so we write

$$ds^2 = a^2\left(Q^2(a)da^2 - dx^2 - dy^2 - dz^2\right)\,.$$

The general solutions reads $Q(a) = (a^b + C)^{f/b}$ with free integration constant C ($C = 0$ gives exact power-law inflation) and m-dependent values b and f: $f = -2 + 1/p$, $b = (4m-5)/(m-1)$. Finally, special solutions for the closed and open Friedmann model are found.

Key words: cosmology — Friedmann models — inflation

AAA subject classification 161

1 Introduction

The de Sitter space-time

$$ds^2 = dt^2 - e^{2Ht}(dx^2 + dy^2 + dz^2)\,, \qquad H \neq 0 \tag{1}$$

is the space-time being mainly discussed to represent the inflationary phase of cosmic evolution. Recently, a space-time defined by

$$ds^2 = dt^2 - |t|^{2p}(dx^2 + dy^2 + dz^2)\,, \qquad p \neq 0 \tag{2}$$

enjoys increasing interest for these discussions, too. Especially, eq. (2) with $p \geq 1$, $t > 0$ is called power-law inflation; and with $p < 0$, $t < 0$ it is called polar inflation.

We summarize some differential-geometrical properties of both de Sitter and power-law/polar inflation in sct. 2 and show, from which kind of scale-invariant field equations they arise in sct. 3; we give the complete set of solutions for the spatially flat and special solutions for the closed and open Friedmann models in closed form for field equations following from the Lagrangian

$$L = R^m \tag{3}$$

in sct. 4, and discuss the results in the final sct. 5 under the point of view that power-law inflation is an attractor solution of the corresponding field equations.

2 Differential–geometrical properties

Eq. (2) defines a self-similar space–time: if we multiply the metric ds^2 by an arbitrary positive constant a^2, then the resulting $d\hat{s}^2 = a^2 ds^2$ is isometric to ds^2. Proof: We perform a coordinate transformation $\hat{t} = at$, $\hat{x} = b(a,p)x$

$\ldots \square$ On the other hand, the de Sitter space-time eq. (1) is not self-similar. Proof: It has a constant non-vanishing curvature scalar. $\square$

Power-law inflation is intrinsically time-oriented. Proof: The gradient of the curvature scalar defines a temporal orientation. $\square$ On the other hand, the expanding ($H > 0$) and the contracting ($H < 0$) de Sitter space-time can be transformed into each other by a coordinate transformation. Proof: Both of them can be transformed to the closed Friedmann universe with scale factor $\cosh(Ht)$, which is an even function of t. $\square$ This is connected with the fact that eq. (2) gives a global description, whereas eq. (1) gives only a proper subset of the full de Sitter space–time.

For $p \to \infty$, eq. (2) tends to eq. (1). Such a statement has to be taken with care, even for the case with real functions. Even more carefully one has to deal with space-times. The most often used limit — the Geroch-limit of space-times — has the property that a symmetry (here: self-similarity) of all the elements of the sequence must also be a symmetry of the limit.

From this it follows that the Geroch limit of space-times (2) with $p \to \infty$ cannot be unique, moreover, it is just the one-parameter set (1) parametrized by arbitrary values $H > 0$.

3 Scale–invariant field equations

A gravitational field equation is called scale-invariant, if to each solution ds^2 and to each positive constant c^2 the resulting homothetically equivalent metric $d\hat{s}^2 = c^2 ds^2$ is also a solution. One example of such field equations is that one following from eq. (3). Moreover, no Lagrangian $L = L(R)$ gives rise to a scale-invariant field equation which is not yet covered by (3) already.

Secondly, for

$$L = R/16\pi G - \frac{1}{2} g^{ij}\Phi_{,i}\Phi_{,j} + V_0 \exp(\mu\Phi) \tag{4}$$

(let be $8\pi G = 1$, henceforth) the homothetic transformation has to be accompanied by a suitable translation of Φ to ensure scale-invariance. For $\mu \neq 0$, the value of V_0 can be normed to 1, 0 or -1.

A third example is the following: for the Kaluza-Klein ansatz

$$dS^2 = ds^2(x^i) + W(x^i)d\tau^2(x^\alpha) \tag{5}$$

($i, j = 0, \ldots 3$; $\alpha, \beta = 4, \ldots N-1$; $A, B = 0 \ldots N-1$) the N-dimensional Einstein equation $R_{AB} = 0$ is scale-invariant. Here, we restrict to the warped product of the $(N-4)$-dimensional internal space $d\tau^2$ with 4–dimensional space–time ds^2.

Let us consider the limits $m \to \infty$ and $m \to 0$ of eq. (3). One gets $L = \exp(R/R_0)$ and $L = \ln(R/R_0)$, resp. Both of them give rise to a field equation which is not scale-invariant: a homothetic transformation changes also the reference value R_0. For the second case this means a change of the cosmological constant. So we have a similar result as before: the limits exist, but they are not unique.

4 Cosmological Friedmann models

We consider the closed and open model in sct. 4.1. and the spatially flat model in sct. 4.2.

4.1 The closed and open models

Let us start with the closed model. We restrict ourselves to a region where one has expansion, so we may use the cosmic scale factor as time-like coordinate:

$$ds^2 = a^2 \left(Q^2(a)da^2 - d\sigma^2 \right) \qquad Q(a) > 0\,, \tag{6}$$

where

$$d\sigma^2 = dr^2 + \sin^2 r d\Omega^2 \,, \qquad d\Omega^2 = d\theta^2 + \sin^2 \theta d\psi^2 \tag{7}$$

is the positively curved 3-space of constant curvature. It holds: if the 00-component of the field equation (here: eq. (10) below) is fulfilled, then all other components are fulfilled, too. Such a statement holds true for all Friedmann models and "almost all sensible field theories". With ansatz (6, 7) we get via

$$R^0_0 = 3(a^{-4}Q^{-2} + a^{-3}Q^{-3}dQ/da) \tag{8}$$

and

$$R = 6(a^{-3}Q^{-3}dQ/da - a^{-2}) \tag{9}$$

the result: the field equation following from the Lagrangian (3) is fulfilled for metric (6, 7), if and only if

$$mRR^0_0 - R^2/2 + 3m(m-1)Q^{-2}a^{-3}dR/da = 0 \tag{10}$$

holds. For $m = 1$ (Einstein's theory), no solution exists. For all other values m, eqs. (8-10) lead to a second order equation for $Q(a)$.

We look now for solutions with vanishing R^0_0, i.e., with (8) we get

$$Q = C/a \,, \qquad C = \text{const.} > 0 \,. \tag{11}$$

By the way, (6,7) is self-similar if and only if eq. (11) holds. From (9, 11) we get

$$R = D/a^2 \,, \qquad D = -6(l + 1/C^2) \,. \tag{12}$$

We insert (11, 12) into (10) and get

$$C = (2m^2 - 2m - 1)^{1/2} \,, \tag{13}$$

which fulfils (11) if

$$m > (1 + \sqrt{3})/2 \quad \text{or} \quad m < (1 - \sqrt{3})/2 \tag{14}$$

holds. Inserting (11,13) into (6) and introducing synchronized coordinates we get as a result: if (14) holds, then

$$ds^2 = dt^2 - t^2 d\sigma^2/(2m^2 - 2m - 1) \tag{15}$$

is a solution of the fourth order field equation following from Lagrangian (3). It is a self-similar solution, and no other self–similar solution describing a closed Friedmann model exists.

For the open Friedmann model all things are analogous, one gets for

$$(1-\sqrt{3})/2 < m < (1+\sqrt{3})/2$$

and with $\sinh r$ instead of $\sin r$ in eq. (7) the only self-similar open solution (which is flat for $m \in \{0,1\}$)

$$ds^2 = dt^2 - t^2 d\sigma^2/(-2m^2 + 2m + 1)\,.$$

4.2 The spatially flat model

The field equation for the spatially flat model can be deduced from that one of a closed model by a limiting procedure as follows: we insert the transformation $r \to \epsilon r$, $a \to a/\epsilon$, $Q \to Q\epsilon^2$ and apply the limit $\epsilon \to 0$ afterwards. One gets via

$$\lim_{\epsilon \to 0} \sin(\epsilon r) = r$$

the metric

$$ds^2 = a^2 \left(Q^2(a) da^2 - dx^2 - dy^2 - dz^2 \right) \tag{16}$$

with unchanged eqs. (8, 10), whereas eq. (9) yields

$$R = 6a^{-3} Q^{-3} dQ/da\,. \tag{17}$$

The trivial solutions are the flat Minkowski space-time and the model with constant value of Q, i.e., $R = 0$ ($m > 1$ only), which is simply Friedmann's radiation model.

Now, we consider only regions with non-vanishing curvature scalar. For the next step we apply the fact that the spatially flat model has one symmetry more than the closed one: the spatial part of the metric is self-similar. In the coordinates (16) this means that each solution $Q(a)$ may be multiplied by an arbitrary constant. To cancel this arbitrariness, we define a new function

$$P(a) = d(\ln Q)/da \, . \tag{18}$$

We insert (8, 17) into (10) and then (18) into the resulting second order equation for Q. We get the first order equation for P

$$0 = m(m-1)dP/da + (m-1)(1-2m)P^2 + m(4-3m)P/a \, . \tag{19}$$

As it must be the case, for $m = 0, 1$, only $P = 0$ is a solution. For $m = 1/2$, $P \sim a^5$ and therefore, $Q = \exp(ca^6)$, c denotes an integration constant. For the other values m we define $z = aP$ as new dependent and $t = \ln a$ as new independent variable.

Eq. (19) then becomes $0 = dz/dt + gz^2 - bz$, $g = l/m - 2$, $b = (4m - 5)/(m-1)$. For $m = 5/4$ we get $z = -5/(6t - c)$ i.e., $P = -5/(6a \ln(a/c))$, $Q = (\ln(a/c))^{-5/6}$. For the other values we get

$$z = f/(\exp(-bt + c) - 1) \, , \quad f = -b/g \, , \quad P = f/(e^c a^{1-b} - a)$$
$$Q = (\pm a^b + c)^{1/g} \, . \tag{20}$$

For $m \to 1/2$ we get $b \to 6$ and $1/g \to \infty$; for $m \to 5/4$ we get $b \to 0$ and $1/g \to -5/6$, so the two special cases could also have been obtained by a limiting procedure from eq. (20).

Metric (16) with (20) can be explicitly written in synchronized coordinates for special examples only, see e.g. BURD and BARROW (1988).

5 Discussion

We have considered scale–invariant field equations. The three examples mentioned in sct. 3 can be transformed into each other by a conformal transformation of the four–dimensional space-time metric. The parameters of eqs. (3) and (4) are related by $\sqrt{3}\,\mu = \sqrt{2}(2-m)/(m-1)$, cf. SCHMIDT (1989), a similar relation exists to the internal space dimension in eq. (5), one has

$$m = 1 + 1/\left\{1 + \sqrt{3}\,[1 + 2/(N-4)]^{\pm 1/2}\right\} ,$$

for details cf. BLEYER and SCHMIDT (1990). The necessary conformal factor is a suitable power of the curvature scalar (3). A further conformal transformation in addition with the field redefinition $\theta = \tanh \Phi$ leads to the conformally coupled scalar field θ in the potential

$$(1+\theta)^{2+\mu}\,(1-\theta)^{2-\mu} ,$$

the conformal factor being $\cosh^4 \Phi$, cf. SCHMIDT (1988). So, equations stemming from quite different physical foundations are seen to be equivalent. We have looked at them from the point of view of self-simi1ar solutions and of limiting processes changing the type of symmetry.

The general solution for the spatially flat Friedmann models in fourth order gravity (3), eqs. (16, 20), can be written for small c in synchronized coordinates as follows

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,, \qquad a(t) = t^p(1 + \epsilon t^{-bp} + O(\epsilon^2))\,.$$

$\epsilon = 0$ gives exact power-law inflation with $p = (m-1)(2m-1)/(2-m)$ and $b = (4m-5)/(m-1)$. We see: in the range $1 < m < 2$, power-law inflation is an attractor solution within the set of spatially flat Friedmann models for $m \geq 5/4$ only. The generalization to polar inflation ($m > 2$, $p < 0$) is similar.

Acknowledgement. Interesting discussions of the topic with L. AMENDOLA, V. MÜLLER, P. TEYSSANDIER, and A. A. STAROBINSKY are gratefully acknowledged.

References

BLEYER, U., SCHMIDT, H.-J.: 1990, Int. J. Mod. Phys. A, in print.

BURD, A., BARROW, J.: 1988, Nuc1. Phys. B **308**, 929.

SCHMIDT, H.-J.: 1988, Phys. Lett. B **214**, 519.

SCHMIDT, H.-J.: 1989, Class. Quant. Grav. **6**, 557.

Received 1989 November 7

In this reprint done with the kind permission of the copyright owner we removed only obvious misprints of the original, which was published in Astronomische Nachrichten: Astron. Nachr. **311** (1990) Nr. 3, pages 165 - 168. The first cited reference appeared in: Int. J. Mod. Phys. A **5** (1990) 4671 - 4676.

Author's address that time:

H.-J. Schmidt, Zentralinstitut für Astrophysik der AdW der DDR, 1591 Potsdam, R.-Luxemburg-Str. 17a